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Lemniscates and Equipotentials

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This paper deals with several generalizations to surfaces in R^n of geometric properties of lemniscates and level curves in the plane. A lemniscate in the plane is defined as a set of points $L\rho : \{\prod_{i=1}^{n} | z - z_i | = \rho^n\}(\rho > 0)$. The lemniscates $L\rho$ can be described (a) as the level curves of polynomials; (b) as the locus of points the product of whose distances from a finite set of fixed points is constant and (c) as the locus of points

$$\sum_{i=1}^{k} \log r_i^{-1} = \text{const} \qquad (r_i = | z - z_i |),$$

the left hand side being a sum of fundamental solution in R^2 of Laplace's equation.

In higher dimensions, the three interpretations corresponding to (a), (b) and (c) are not equivalent any longer.

The lemniscate surface in R^n based on definition (a), as a level surfaces of a polynomial in n real variables, was introduced by Nagy [2] in 1950, and the idea was extended to level surfaces of rational functions by Schurer [4].

The concept of a lemniscate based on interpretation (c) was first generalized to R^3 by Polya and Szego [3] who defined a lemniscate surface as a locus $\sum_{k=1}^{n} r_k^{-1} = \text{cont.}$, r_k being again the distance of the variable point from a fixed one. Our definition of a lemniscate surface in R^n, corresponding to definition (c) in the plane, will be $\sum_{k=1}^{p} m_k r_k^{2-n} = \text{const}$, $m_k > 0, n \geqslant 3$, with the same meaning of r_k. Negative values of m_k are also considered.

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Both interpretations (b) and (c) are special cases of equipotential surfaces to be studied in this paper: these are defined as follows:

Let E_1 and E_2 be compact, convex sets in R^n, $E_1 \cap E_2 = \varnothing$; μ a positive bounded measure defined on R^n with support on $E_1 \cup E_2$. Let $\Phi(r)$ be a decreasing function on $(0, \infty)$; $\Phi(r) \in C^2$. Let x_0 denote a point of $R^n - (E_1 \cup E_2)$. Consider the potentials $V(x_0) = \int_{E_1} \Phi(r_1)\, d\mu$ and

$$U(x_0) = \int_{E_1} \Phi(r_1)\, d\mu - \int_{E_2} \Phi(r_2)\, d\mu.$$

Here r_i denotes the distance from x_0 to x_i, $x_i \in E_i$, $i = 1, 2$. We set

$$V_\lambda = \{x_0 \mid V(x_0) = \lambda\}, \qquad U_\lambda = \{x_0 \mid U(x_0) = \lambda\}.$$

The surfaces V_λ, in this general form, were first investigated by Kahane [1]. In the present paper generalizations to the surfaces V_λ and U_λ are given of geometric inequalities derived by the author for level curves of Green's function in the plane [5, 7] and rational functions [6].

Our first result is a generalization of a theorem on the bisector of a chord of a level curve of Green's function [5, Theorem 1].

THEOREM 1. *Let M_1, M_2 be (distinct) points of some V_λ. Denote the mid-point of the segment M_1M_2 by M. Then the hyperplane Π through M, normal to the segment M_1M_2, must intersect E_1.*

Proof. If we assume M_1 and all points of E_1 lie in the same half space defined by Π, that is, the distance r_{M_1P} of M_1 from P is smaller than r_{M_2P} for all $P \in E_1$, then $\Phi(r_{M_1P}) > \Phi(r_{M_2P})$ and $V(M_1) > V(M_2)$; thus we arrive at a contradiction.

COROLLARY. *Each normal to V_λ must intersect E_1.*

This was proved as an independent theorem by Kahane [1, Theorem 1].

THEOREM 2. *Let M_1, M_2 be distinct points of some U_λ, and let $M = \frac{1}{2}(M_1 + M_2)$, and Π be the hyperplane through M, normal to M_1M_2. Then Π cannot separate E_1 and E_2.*

Proof. Assume the contrary, so that e.g., $\Phi(r_{M_1P_1}) > \Phi(r_{M_2P_1})$ for all $P_1 \in E_1$ and $\Phi(r_{M_1P_2}) < \Phi(r_{M_2P_2})$ for all $P_2 \in E_2$. This implies $U(M_1) > U(M_2)$.

COROLLARY. *Let N denote a normal to some U_λ. A hyperplane through point M_2 on U_λ, containing N, cannot separate E_1 and E_2.*

The proof follows by using Theorem 2 and letting M_1 approach M_2.

THEOREM 3. *Assume that the set E_1 is contained in a half space S_1 bounded by a hyperplane π. Let S_2 be the complimentary half space $R^n - S_1$. Then a normal q to π can contain at most one point of $V_\lambda \cap S_2$, for any equipotential V_λ.*

Proof. Assume there exist distinct points M_1, M_2 lying on $V_\lambda \cap S_2 \cap q$. Then the hyperplane through their midpoint orthogonal to q will be parallel to π and cannot intersect E_1. Theorem 1 is contradicted.

THEOREM 4. *Let E_1 be contained in a ball B, with center Ω. Then any surface V_λ lying outside of B is star-shaped with respect to Ω.*

Proof. Let P be an arbitrary point of V_λ. Then the hyperplane tangent to B and orthogonal to the ray from Ω to P separates E from V_λ, and by Theorem 2 it cannot have a second intersection with V_λ.

THEOREM 5. *Let E_1 be contained in a ball B of radius a. Let some V_λ intersect a concentric sphere of radius λa, $\lambda \geqslant 1$; then V_λ can intersect no concentric sphere of radius greater than $(\lambda + 2)a$.*

The proof is similar to that of Theorem 1 in [5, p. 61].

The next theorem furnishes a bound on the radius of curvature of curves lying on the U_λ. This result generalizes those of Theorem 3 in [6]. The corresponding theorem for the surfaces V_λ is due to Kahane [1, Theorem 3].

THEOREM 6. *Let the surfaces U_λ be defined as above. Let the function $\Phi(r)$ satisfy the inequalities*

$$0 < r\Phi''(r) \leqslant -(\alpha + 1)\,\Phi'(r), \quad \alpha \geqslant 0. \tag{1}$$

Let E_1 be contained in a half space S_1 bounded by a hyperplane π_1, and let E_2 be contained in a half space S_2 bounded by a hyperplane π_2 parallel to π_1; $S_1 \cap S_2 = \varnothing$. Let P be an arbitrary point in $R^n - (S_1 \cup S_2)$. Denote by N the (vector) normal to U_λ at P. Let $A_1 = N \cap \pi_1$, and $A_2 = N \cap \pi_2$. Let C be a curve lying on U_λ with N as a normal.

Then the radius of curvature of C is numerically greater than

$$(\alpha + 1)^{-1} \min(r_{PA_1}, r_{PA_2}).$$

Proof. Write $\Phi(r) = \varphi(r^2)$. Then

$$N = \operatorname{grad} U = 2\int_{E_1} (x - x_1)\,\varphi'(r_1{}^2)\,d\mu - 2\int_{E_2} (x - x_2)\,\varphi'(r_2{}^2)\,d\mu.$$

Orient the space so that N has the direction of the nth coordinate axis, and P is the origin. Suppose S_1 is the half space

$$x^n > a_1 x^1 + \cdots a_{n-1} x^{n-1} + b,$$

and S_2 is the half space

$$x^n < a_1 x^1 + \cdots a_{n-1} x^{n-1} - B,$$

where $(x^1, x^2 \cdots x^n)$ is the variable point and b, B are some positive constants.

For our choice of the coordinate system,

$$\int -x_1{}^i \varphi'(r_1{}^2)\, d\mu - \int -x_2{}^i \varphi'(r_2{}^2)\, d\mu = 0 \qquad \text{for} \quad i = 1, 2, \ldots, n-1$$

where $x_i = (x_i{}^1, x_i{}^2, \ldots, x_i{}^n)$, $i = 1, 2$, and

$$\| N \| = N^n > 2 \int b(-\varphi'(r_1))\, d\mu + 2B \int (-\varphi'(r_2))\, d\mu > 0.$$

Let $\dot{x} = dx/ds$ be the unit vector tangent at P to the curve C, where s denotes arc length on C. Then

$$-\tfrac{1}{2}\dot{x} \cdot \dot{N} = \int -\varphi'(r_1{}^2)\, d\mu - \int -\varphi'(r_2{}^2)\, d\mu$$
$$- 2 \left[\int (x_1 \cdot \dot{x})^2\, \varphi''(r_1{}^2)\, d\mu - \int (x_2 \cdot \dot{x})^2\, \varphi''(r_2{}^2)\, d\mu \right]. \qquad (2)$$

We may assume $b < B$.

There are two cases.

Case 1. $-\dot{x} \cdot \dot{N} > 0$. We have to prove

$$\ddot{x} = -(\dot{x} \cdot \dot{N} / \| N \|) < (\alpha + 1)/b. \qquad (3)$$

By substitution of (2) into (3) we obtain

$$\alpha \int -\varphi'(r_1{}^2)\, d\mu + \left\{ \frac{B}{b}(1 + \alpha) + 1 \right\} \int -\varphi'(r_2{}^2)\, d\mu$$
$$+ 2 \int (x_1 \cdot \dot{x})^2\, \varphi''(r_1{}^2)\, d\mu - 2 \int (x_2 \cdot \dot{x})^2\, \varphi''(r_2{}^2)\, d\mu > 0. \qquad (4)$$

The first and third terms in (4) are positive. The inequality (3) would follow if

$$\int [(2 + \alpha)(-\varphi'(r_2{}^2)) - 2(x_2 \cdot \dot{x})^2\, \varphi''(r_2{}^2)]\, d\mu > 0.$$

Hypothesis (1) implies that $\varphi''(r^2) \leqslant -[(\alpha + 2)\, \varphi'(r^2)/2r^2]$. Since $(x_2 \cdot \dot{x})^2 \leqslant r_2^2$, the result (4) follows.

Case 2. $\ddot{x} < 0$. This case can be treated similarly to Case 1.

Remark. The inequality (1), with $\alpha = 0$, is satisfied by the logarithmic potential and also (for $\alpha > 0$) by the potential $\varnothing(r) = r^{-\alpha}$.

The following Theorems 7 and 8 will be proved for the specialized case where $\Phi(r) \equiv \log(1/r)$ and μ is a discrete point measure. This corresponds to the extension of the definition of a lemniscate based on interpretation (b), that is,

$$V_\lambda = \left\{ x \;\middle|\; \sum_{j=1} -m_j \log r_j = \lambda \right\}, \qquad m_j > 0. \tag{5}$$

The first theorem represents an extension of Walsh's theorem [9, p. 13] to n dimensions. In the proof we shall require the following coincidence lemma:

LEMMA 1. *Let $x_1, x_2, \ldots, x_n$ lie in a ball B in R^n, and let $m_1, m_2, \ldots, m_n$ be positive. Then for every $x \in R^n - B$ there exists an $x_0 \in B$ such that*

$$\frac{1}{M} \sum_{k=1}^{n} \frac{m_k(x - x_k)}{\| x - x_k \|^2} = \frac{x - x_0}{\| x - x_0 \|^2},$$

where $M = \sum_{k=1}^{n} m_k$.

Proof. Inversion in a sphere with center x transforms B one-to-one onto a ball B^1.

The image ξ_k of $x_k \in B$ is given by

$$\xi_k = - \frac{x - x_k}{\| x - x_k \|^2} + x.$$

Now $M^{-1} \sum m_k \xi_k$, the center of gravity of the ξ_k, lies in B^1; therefore its antecedent $x_0 \in B$.

Our extension of Walsh's Theorem deals with the critical points of a lemniscate surface

$$V_\lambda = \{x \mid V(x) = \lambda\} \tag{6}$$

where

$$V(x) \equiv \sum_{k=1}^{p} m_k \log[1/\| x - x_k \|] + \sum_{i=1}^{q} \mu_i \log[1/\| x - y_i \|]$$

where the fixed points x_k lie in a ball B_1 and the fixed points y_i in a ball B_2.

THEOREM 7. *All the critical points of the lemniscate surface* (6) (i.e., *the points of* (6) *for which* grad $V = 0$) *lie in* B_1, B_2 *and a third ball with center* $(M_2a_1 + M_1a_2)/(M_1 + M_2)$ *and radius* $(M_2r_1 + M_1r_2)/(M_1 + M_2)$; a_1 *and* a_2 *denote the centers of* B_1 *and* B_2, *respectively*, r_1, r_2 *denote their radii*,

$$M_1 = \sum_{k=1}^{p} m_k, \qquad M_2 = \sum_{i=1}^{q} \mu_i .$$

Proof. The critical points of (6) satisfy

$$\sum_{k=1}^{p} m_k \frac{x - x_k}{r_k^2} + \sum_{i=1}^{q} \mu_i \frac{x - y_i}{r_i^2} = 0.$$

The proof of the theorem follows by application of the coincidence lemma for both B_1 and B_2, and by considerations similar to the proof of the original theorem in the plane [9, pp. 13–15].

The next theorem generalizes Corollary 1 of Theorem 4 in [5] which gives sharp bounds for the curvature of lemniscates and level curves of Green's function.

THEOREM 8. *Let* E_1 *be contained in a ball of radius a. Then the surface* (5) *is convex if it lies outside of the concentric ball of radius* $\sqrt{2}a$.

Proof. The notation used will be the same as in Theorem 6. We obtain for the normal N to V_λ at the point x:

$$\text{grad } V_\lambda = N = -\sum_{j=1}^{n} m_j \frac{x - x_j}{r_j^2} \tag{7}$$

Here $x_1, x_2, ..., x_n$ denote points in E_1.

Let C be any curve on V_λ whose normal at x has the direction of N or of $-N$. We shall prove that necessarily $\ddot{x}$ has there the direction of N. The convexity of V then follows. Since $N \cdot \dot{x} = 0$ along C, we have there

$$\dot{N} \cdot \dot{x} + N \cdot \ddot{x} = 0.$$

We shall prove that at x, $-\dot{N} \cdot \dot{x} \geqslant 0$. By differentiation of (7), we obtain

$$-\dot{N} \cdot \dot{x} = \sum_{j=1}^{n} \frac{m_j}{r_j^2} - 2 \sum_{j=1}^{n} \frac{m_j}{r_j^4} (x - x_j \cdot \dot{x})^2 \tag{8}$$

with the auxiliary condition

$$N \cdot \dot{x} = \sum_{j=1}^{n} \frac{m_j}{r_j^2} (x - x_j \cdot \dot{x}) = 0. \tag{9}$$

Let the fixed points x_j be divided into two classes:

$$x_{1j} \text{ denote } x_j \text{ for which } (x - x_j) \cdot \dot{x} \geqslant 0,$$
$$x_{2j} \text{ denote } x_j \text{ for which } (x - x_j) \cdot \dot{x} < 0.$$

The sum (7) can be rearranged as follows:

$$N = \sum_{i=1}^{k} M_i n_i , \tag{10}$$

where

$$n_i \cdot \dot{x} = 0, \qquad \sum_{i=1}^{k} M_i = \sum_{j=1}^{n} m_j , \qquad n_i = l_i \frac{x_{1i} - x}{r_{1i}^2} + (1 - l_i) \frac{x_{2i} - x}{r_{2i}^2} ,$$
$$1 \geqslant l_i \geqslant 0, \quad i = 1, 2, ..., k. \tag{11}$$

These conditions imply that n_i lies in the plane containing x, x_{1i} and x_{2i} and is orthogonal to $\dot{x}$. We will prove that $-\dot{n}_i \cdot \dot{x} \geqslant 0$ for all i.

By differentiation of n_i in (11) and inner multiplication by $\dot{x}$ we obtain (the subscript i is dropped for simplicity):

$$\begin{aligned} -\dot{n} \cdot \dot{x} &= \| n \|^2 + \left\| n - \frac{x_1 - x}{r_1^2} \right\| \left\| n - \frac{x_2 - x}{r_2^2} \right\| \\ &\quad - \left[\left(\frac{x_1 - x}{r_1^2} - n \right) \cdot \dot{x} \right] \left[\left(n - \frac{x_2 - x}{r_1^2} \right) \cdot \dot{x} \right] \\ &\geqslant \| n \|^2 - \left\| n - \frac{x_1 - x}{r_1^2} \right\| \left\| n - \frac{x_2 - x}{r_2^2} \right\| . \end{aligned} \tag{12}$$

Condition (12) is equivalent to the geometric condition that the angle between the vectors $x_1 - x$ and $x_2 - x$ is acute; this condition is assured by the hypothesis of the theorem. It follows that $n_i \cdot \ddot{x}$ is positive for all pairs of points x_{1i} and x_{2i} in E, and therefore $N \cdot \ddot{x}$ is positive and the curve C is convex.

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